

# Multilinear algebra in the context of diffeology

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## Abstract

This note is dedicated to some details of multilinear algebra on diffeological vector spaces; most of them are the to-be-expected corollaries of standard constructions and various facts of diffeology collected elsewhere. Most of the attention is paid to the implications of the notion of the diffeological dual (introduced elsewhere) and of its differences with respect to the standard notion. We follow whenever possible a rather naive approach, to supplement the existing works on the matter.

**Keywords:** diffeology, diffeological vector space, diffeological dual

MSC (2010): 53C15, 15A69 (primary), 57R35, 57R45 (secondary).

## Introduction

The aim of this work is rather modest; it is to render explicit what becomes of some basic facts of multilinear algebra when these are considered for *diffeological vector spaces*. The main (and most easily accessible) reference for the subject of diffeology in general, and diffeological vector spaces in particular, is the excellent and comprehensive source [3]. Our work here builds a lot on the definitions and facts already presented in [7], and previously in [6] (see also [2]). The main notions, such as those of the diffeological dual and the tensor product, were announced, in a definitive manner, in [7], where the discussion is rather concise; part of our intention is to render explicit what is implicit there, and to provide specific motivations stemming from various examples.

Our main focus, and the reason for revisiting the (essentially) elementary subject of multilinear algebra, is the somewhat surprising fact that even for finite-dimensional diffeological vector spaces not all (multi)linear maps between them are smooth. Thus, it may, or may not, be obvious whether the classical isomorphisms of multilinear algebra continue to exist in the diffeological context, in the sense whether their restrictions to the smooth subspaces are well-defined and, if so, whether these restrictions are diffeomorphisms in their turn. These are the kinds of questions that we consider below, in addition to providing a few explicit proofs to the statements announced or implicit in [7].

Finally, note that diffeological vector spaces appear naturally in the context of various attempts to construct tangent bundles to diffeological spaces, see [1] and references therein.

**The structure** In the first section we collect the necessary notions regarding *diffeological structures* and *diffeological vector spaces*. In the other three sections we consider some of the classical isomorphisms (and discuss some typical constructions) of multilinear algebra from the diffeological point of view.

**Acknowledgments** I came across the notion of diffeology only recently (but I do have to thank xxx.lanl.gov for that), but the time I've spent wondering the ways of mathematical research will soon constitute a half of my total lifetime. This might in part explain why I feel this work owes a lot to a colleague of mine who is not a mathematician and who, at the time of these words being written, has no idea that it exists. The name of my colleague is Prof. Riccardo Zucchi; what this work owes to him is inspiration and motivation, first of all those needed to be brave enough as to invest time and effort into something destined to be imperfect, and to do what, correctly or not, seems right or necessary at one particular moment only, independently of what the next step would be. Finally, it is my pleasure to thank Prof. Dan Christensen and Prof. Enxin Wu for providing extremely useful comments on the first version of this paper, and the anonymous referee for same.

# 1 Diffeological spaces and diffeological vector spaces

Here we recall as briefly as possible the main notions regarding diffeology ([4], [5]) and diffeological (vector) spaces; more details can be found in [3]; see [2], [6], [7] for diffeological vector spaces.

**Diffeological spaces** A **diffeological space** (see [5]) is a pair  $(X, \mathcal{D}_X)$  where  $X$  is a set and  $\mathcal{D}_X$  is a specified collection of maps  $U \rightarrow X$  (called **plots**) for each open set  $U$  in  $\mathbb{R}^n$  and for each  $n \in \mathbb{N}$ , such that for all open subsets  $U \subseteq \mathbb{R}^n$  and  $V \subseteq \mathbb{R}^m$  the following three conditions are satisfied:

1. (The covering condition) Every constant map  $U \rightarrow X$  is a plot;
2. (The smooth compatibility condition) If  $U \rightarrow X$  is a plot and  $V \rightarrow U$  is a smooth map (in the usual sense) then the composition  $V \rightarrow U \rightarrow X$  is also a plot;
3. (The sheaf condition) If  $U = \cup_i U_i$  is an open cover and  $U \rightarrow X$  is a set map such that each restriction  $U_i \rightarrow X$  is a plot then the entire map  $U \rightarrow X$  is a plot as well.

Usually, one just writes  $X$  to denote a diffeological space; a standard example of a diffeological space is a smooth manifold  $M^n$ , with the diffeology given by all smooth maps of form  $U \rightarrow M^n$ , for  $U$  a domain in some  $\mathbb{R}^k$ . If we have two diffeological spaces,  $X$  and  $Y$ , and a set map  $f : X \rightarrow Y$  between them, this map is said to be **smooth** if for every plot  $p : U \rightarrow X$  of  $X$  the composition  $f \circ p$  is a plot of  $Y$ .

**Comparing diffeologies** Given a set  $X$ , the set of all possible diffeologies on  $X$  is partially ordered by inclusion: a diffeology  $\mathcal{D}$  on  $X$  is said to be **finer** than another diffeology  $\mathcal{D}'$  if  $\mathcal{D} \subset \mathcal{D}'$  (whereas  $\mathcal{D}'$  is said to be **coarser** than  $\mathcal{D}$ ).

**Generated diffeology and quotient diffeology** These are two (out of many) ways to construct a diffeology; we will use both. If  $X$  is a set and we are given a set of maps  $A = \{U_i \rightarrow X\}_{i \in I}$ , the **diffeology generated by  $A$**  is the smallest, with respect to inclusion, diffeology on  $X$  that contains  $A$ ; its plots all locally factor through those of  $A$ .

If now  $X$  is a diffeological space, let  $\sim$  be an equivalence relation on  $X$ , and let  $\pi : X \rightarrow Y := X/\sim$  be the quotient map. The **quotient diffeology** ([3]) on  $Y$  is the diffeology in which  $p : U \rightarrow Y$  is the diffeology in which  $p : U \rightarrow Y$  is a plot if and only if each point in  $U$  has a neighbourhood  $V \subset U$  and a plot  $\tilde{p} : V \rightarrow X$  such that  $p|_V = \pi \circ \tilde{p}$ .

**Subset diffeology** Let  $X$  be a diffeological space, and let  $Y \subseteq X$  be its subset. The **subset diffeology** on  $Y$  is the coarsest diffeology on  $Y$  making the inclusion map  $Y \hookrightarrow X$  smooth.

**Pushforwards and pullbacks of a diffeology** For any diffeological space  $X$ , any set  $X'$ , and any map  $f : X \rightarrow X'$  there exists a finest diffeology on  $X'$  that makes the map  $f$  smooth; it is called the **pushforward of the diffeology of  $X$  by the map  $f$** . If now we have a map  $f : X' \rightarrow X$  then the **pullback** of the diffeology of  $X$  by the map  $f$  is the coarsest diffeology on  $X'$  such that  $f$  is smooth.

**The diffeological direct product** Let  $\{X_i\}_{i \in I}$  be a collection of diffeological spaces. The **product diffeology** on the direct product  $X = \prod_{i \in I} X_i$  is the coarsest diffeology such that for each index  $i \in I$  the natural projection  $\pi_i : \prod_{i \in I} X_i \rightarrow X_i$  is smooth.

**Functional diffeology** Let  $X, Y$  be two diffeological spaces, and let  $C^\infty(X, Y)$  be the set of smooth maps from  $X$  to  $Y$ . Let  $\text{EV}$  be the *evaluation map*, defined by

$$\text{EV} : C^\infty(X, Y) \times X \rightarrow Y \text{ and } \text{EV}(f, x) = f(x).$$

The **functional diffeology** on  $C^\infty(X, Y)$  is the coarsest diffeology such that the evaluation map is smooth.

**Diffeological vector spaces** Let  $V$  be a vector space over  $\mathbb{R}$ . The **vector space diffeology** on  $V$  is any diffeology of  $V$  such that the addition and the scalar multiplication are smooth, that is,

$$[(u, v) \mapsto u + v] \in C^\infty(V \times V, V) \text{ and } [(\lambda, v) \mapsto \lambda v] \in C^\infty(\mathbb{R} \times V, V),$$

where  $V \times V$  and  $\mathbb{R} \times V$  are equipped with the product diffeology; equipped with a vector space diffeology,  $V$  is called a **diffeological vector space**.

**Smooth linear maps, subspaces and quotients** Given two diffeological vector spaces  $V$  and  $W$ , the space of **smooth linear maps** between them is denoted by  $L^\infty(V, W) = L(V, W) \cap C^\infty(V, W)$ . This is an  $\mathbb{R}$ -linear subspace of  $L(V, W)$ , frequently a proper subspace. A **subspace** of a diffeological vector space  $V$  is any vector subspace of  $V$  endowed with the subset diffeology (and is a diffeological vector space on its own). Finally, if  $V$  is a diffeological vector space and  $W \leq V$  is a subspace of it then the quotient  $V/W$  is a diffeological vector space with respect to the quotient diffeology.

**Direct sum of diffeological vector spaces** Let  $V_1, \dots, V_n$  be a collection of diffeological vector spaces. The usual direct sum  $V_1 \oplus \dots \oplus V_n$  of these spaces, equipped with the *product* diffeology, is a diffeological vector space.

**Fine diffeology on vector spaces** The **fine diffeology** on a vector space  $\mathbb{R}$  is the *finest* vector space diffeology on it; endowed with such,  $V$  is called a *fine vector space*. Note that *any* linear map between two fine vector spaces is smooth. In the finite-dimensional case, the fine spaces are precisely the spaces  $\mathbb{R}^n$  with their standard diffeology (one consisting of all usually smooth maps into them).

**Diffeological dual** For a diffeological vector space  $V$ , its **diffeological dual**  $V^*$  ([6], [7]) is defined as the set of all smooth linear maps  $V \rightarrow \mathbb{R}$  into the standard  $\mathbb{R}$ , endowed with the functional diffeology.

**The tensor product** Given a finite collection  $V_1, \dots, V_n$  of diffeological vector spaces, their usual tensor product  $V_1 \otimes \dots \otimes V_n$  is endowed with the **tensor product diffeology** (see [6], [7]) defined as the quotient diffeology corresponding to the usual representation of  $V_1 \otimes \dots \otimes V_n$  as the quotient of  $V_1 \times \dots \times V_n$  (endowed with the product diffeology) by the kernel of the universal map onto  $V_1 \otimes \dots \otimes V_n$ . In other words, the tensor product diffeology is the pushforward of the product diffeology on  $V_1 \times \dots \times V_n$  by the universal map. The diffeological tensor product possesses the usual universal property by Theorem 2.3.5 of [6], namely, if  $W$  is another diffeological vector space then the space of all smooth linear maps  $V_1 \otimes \dots \otimes V_n \rightarrow W$ , considered with the functional diffeology, is diffeomorphic to the space of all smooth multilinear maps  $V_1 \times \dots \times V_n \rightarrow W$  (also considered with the functional diffeology).

## 2 Smooth linear and bilinear maps

In this section we consider some of the classic isomorphisms of multilinear algebra, showing that they do extend into the context of diffeology, while providing examples that show that the *a priori* difference is significant.

### 2.1 Linear maps and smooth linear maps

The sometimes significant difference just mentioned is illustrated by the following example; note that the existence of such examples has already been mentioned in [7], see Example 3.11.

**Example 2.1.** *This is an example of  $V$  such that  $L^\infty(V, \mathbb{R}) < L(V, \mathbb{R})$ . Let  $V = \mathbb{R}^n$  equipped with the coarse diffeology; we claim that the only smooth linear map  $V \rightarrow \mathbb{R}$  is the zero map. Indeed, let  $f : V \rightarrow \mathbb{R}$  be a linear map; it is smooth if and only if, for any plot  $p$  of  $V$ , the composition  $f \circ p$  is a plot of  $\mathbb{R}$ , i.e.,  $f \circ p$  is a usual smooth map  $U \rightarrow \mathbb{R}$  for some domain  $U \subseteq \mathbb{R}^k$ . However, by definition of the coarse diffeology,  $p$  is allowed to be any set map  $U \rightarrow V$ , so it may not even be continuous.*

To provide a specific instance, choose some basis  $\{v_1, \dots, v_n\}$  of  $V = \mathbb{R}^n$ , and a basis  $\{v\}$  of  $\mathbb{R}$ .<sup>1</sup> With respect to these,  $f$  is given by  $n$  real numbers, more precisely, by the matrix  $(a_1 \dots a_n)$ . Consider the following  $n$  plots  $p_i$  of  $V$  with  $i = 1, \dots, n$ , defined by setting  $p_i : \mathbb{R} \rightarrow V$  and  $p_i(x) = |x|v_i$ ; then  $(f \circ p_i)(x) = a_i|x|v$ . The only way for this latter map to be smooth is to have  $a_i = 0$ , and this must hold for  $i = 1, \dots, n$ , whence our claim.

The example just given shows that the *a priori* issue of there being diffeological vector spaces  $V$ ,  $W$  such that  $L^\infty(V, W) < L(V, W)$  does indeed occur, and rather easily. True, this requires some rather surprising vector spaces/diffeologies for this happen; but diffeology was designed for dealing with surprising, or at least unusual from the Differential Geometry point of view, objects,<sup>2</sup> and furthermore, the very essence of what diffeology aims to add to the ‘standard’ differential setting is the flexibility of what can be called smooth. In particular, the fact that *any* given map  $\mathbb{R}^k \supseteq U \rightarrow X$  can be a plot for some diffeology on a given set  $X$  (for instance, for the diffeology generated by this map) easily gives rise to some surprising spaces.

**Example 2.2.** Once again, consider  $V = \mathbb{R}^n$  and some basis  $\{v_1, \dots, v_n\}$  of  $V$ ; endow it with the vector space diffeology generated by all smooth maps plus the map  $p_i$  already mentioned, that is, the map  $p_i : \mathbb{R} \rightarrow V$  acting by  $p_i(x) = |x|v_i$ . Let  $v$  be a generator of  $\mathbb{R}$  (i.e., any non-zero vector). Using the same reasoning as in Example 2.1, one can show that if  $f : V \rightarrow \mathbb{R}$  is linear and, with respect to the bases chosen has matrix  $(a_1 \dots a_n)$ , then for it to be smooth we must have  $a_i = 0$ ; hence the (usual vector space) dimension  $L^\infty(V, \mathbb{R})$  (i.e., that of the diffeological dual of  $V$ ) is at most  $n - 1$ .<sup>3</sup>

This reasoning can be further extended by choosing some natural number  $1 < k < n$  and a set of  $k$  indices  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ , and endowing  $V$  with the vector space diffeology generated by all smooth maps plus the set  $\{p_{i_1}, \dots, p_{i_k}\}$ . Arguing as above, we can easily conclude that  $\dim(L^\infty(V, \mathbb{R}))$  is at most  $n - k$ .

The examples just cited show, in particular, that the diffeological dual (defined in [6] and [7]) of a diffeological vector space can be much different from the usual vector space dual. Given the importance of the isomorphism-by-duality in multilinear algebra’s arguments, the implications of this difference deserve to be considered.

## 2.2 Bilinear maps and smooth bilinear maps

In this section we consider the same issues as above in the case of bilinear maps: given two diffeological vector spaces, what is the difference between the set of all bilinear maps on one of them with values in the other, and the set of all such bilinear maps that in addition are smooth?

**Smooth bilinear maps** Let  $V, W$  be two diffeological spaces. As usual, a  $W$ -valued bilinear map is a map  $V \times V \rightarrow W$  linear in each argument; it is considered to be, or not, smooth with respect to the product diffeology on  $V \times V$  and the diffeology on  $W$ . We first illustrate that what happens for linear maps does (expectedly) happen for bilinear maps, i.e., the set of smooth bilinear maps can be strictly smaller than that of bilinear maps.

Our notation is as follows. Given  $V, W$  two diffeological vector spaces, let  $B(V, W)$  be the set of bilinear maps on  $V$  with values in  $W$ , and let  $B^\infty(V, W)$  be the set of those bilinear maps that are smooth with respect to the product diffeology on  $V \times V$  and the given diffeology on  $W$ .

**Example 2.3.** The examples seen in the previous section provide readily the instances of  $V$  and  $W$  such that  $B^\infty(V, W)$  is a proper subspace of  $B(V, W)$ . Indeed, let us take  $V = \mathbb{R}^n$  equipped with the coarse diffeology, and let  $W = \mathbb{R}$  considered with the standard diffeology. It is easy to extend the reasoning of Example 2.1 to show that for these two spaces  $B^\infty(V, W) = 0$ .

Once again, take a basis  $\{v_1, \dots, v_n\}$  of  $V$  and a basis  $\{w\}$  of  $W$ ; let  $f \in B^\infty(V, W)$ . Then with respect to the bases chosen  $f$  is defined by the matrix  $(a_{ij})_{n \times n}$  where  $f(v_i, v_j) = a_{ij}w$ . For each  $i = 1, \dots, n$

<sup>1</sup>Obviously, the respective canonical bases would do the job just fine.

<sup>2</sup>This is a story beautifully told in the Preface and Afterword to the excellent book [3].

<sup>3</sup>One can actually show that  $f$  is smooth if and only if  $a_i = 0$ , and so  $\dim(L^\infty(V, \mathbb{R}))$  is precisely  $n - 1$ ; we do not elaborate on this, since we mostly interested in showing that it can be strictly smaller (by any admissible value, as we see below).

consider the already-seen map  $p_i : \mathbb{R} \rightarrow V$  given by  $p_i(x) = |x|v_i$ ; this map is a plot of  $V$  by definition of the coarse diffeology (that includes all set maps from domains of various  $\mathbb{R}^k$  to  $V$ ). Now call  $p_{ij}$  the product map  $p_{ij} : \mathbb{R} \rightarrow V \times V$ , i.e. the map given by  $p_{ij}(x) = (p_i(x), p_j(x))$ ; it is obviously a plot for the product diffeology on  $V \times V$ . Putting everything together, we get that  $(f \circ p_{ij})(x) = a_{ij}|x|w$ ; since this must be a plot of  $\mathbb{R}$ , i.e. smooth in the usual sense, we get that  $a_{ij} = 0$ . The indices  $i, j$  being arbitrary, we conclude that the only way for  $f$  to be smooth is for it to be the zero map, whence the conclusion.

The example just given stresses the importance of making a distinction between bilinear maps and smooth bilinear maps, showing that the two spaces can be (*a priori*) quite different, and motivates the next paragraph.

**The function spaces  $B^\infty(V, W)$  and  $L^\infty(V, L^\infty(V, W))$**  As is well-known, in the usual setting each bilinear map can be viewed as a linear map  $V \rightarrow L(V, W)$ . In the diffeological context, since *a priori* we might have  $L^\infty(V, W) < L(V, W)$ , we need to consider the question of whether any *smooth* bilinear map can be seen as a *smooth* map  $V \rightarrow L^\infty(V, W)$ , where the latter is endowed with the functional diffeology. By the properties of the diffeologies involved, the answer is in the affirmative, as we now show.

**Lemma 2.4.** *Let  $V, W$  be two diffeological vector spaces, let  $f : V \times V \rightarrow W$  be a bilinear map smooth with respect to the product diffeology on  $V \times V$  and the given diffeology on  $W$ , and let  $G : V \rightarrow L^\infty(V, W)$  be a linear map that is smooth with respect to the given diffeology on  $V$  and the functional diffeology on  $L^\infty(V, W)$ . Then:*

- *for every  $v \in V$  the linear map  $F(v) : V \rightarrow W$  given by  $F(v)(v') = f(v, v')$  is smooth;*
- *the bilinear map  $g : V \times V \rightarrow W$  given by  $g(v, v') = G(v)(v')$  is smooth.*

*Proof.* Let us prove the first statement. Fix a  $v \in V$ ; to show that  $F(v)$  is smooth, we need to show that for every plot  $p : U \rightarrow V$  the composition  $F(v) \circ p$  is a plot of  $W$ . Fixing an arbitrary plot  $p : U \rightarrow V$  of  $V$ , we define  $\tilde{p} : U \rightarrow V \times V$  by setting  $\tilde{p}(x) = (v, p(x))$  for all  $x \in U$ ; this is indeed a plot for the product diffeology, since the projections on both factors are smooth:  $\pi_1 \circ \tilde{p}$  is a constant map in  $V$ , while  $\pi_2 \circ \tilde{p} = p$ , which is a plot by assumption. We thus obtain  $F(v) \circ p = f \circ \tilde{p}$ ; the latter map is a plot of  $W$  since  $f$  is smooth; and since  $p$  is arbitrary, so is  $F(v)$ .

To prove the second statement, it suffices to observe that  $g$  writes as the composition  $g = \text{EV} \circ (G \times \text{Id}_V)$ ; the map  $\text{Id}_V$  being obviously smooth,  $G$  being smooth by assumption, their product being smooth by definition of the product diffeology, and, finally, the evaluation map  $\text{EV}$  being smooth by the definition of the functional diffeology, we get the conclusion.  $\square$

What the above lemma gives us are the following two maps:

- the map  $\tilde{F} : B^\infty(V, W) \rightarrow L(V, L^\infty(V, W))$  that assigns to each  $f \in B^\infty(V, W)$  the map  $F$  of the lemma (i.e., the specified map that to each  $v \in V$  assigns the smooth linear map  $F(v) : V \rightarrow W$ ). Observe that  $F$  now writes as  $F = \tilde{F}(f)$  and that the following relation holds:  $f = \text{EV} \circ (F \times \text{Id}_V)$ ;
- the map  $\tilde{G} : L^\infty(V, L^\infty(V, W)) \rightarrow B^\infty(V, W)$  that assigns to each  $G \in L^\infty(V, L^\infty(V, W))$  the map  $g = \text{EV} \circ (G \times \text{Id}_V)$ . This latter map now writes as  $g = \tilde{G}(G)$ .

Before going further, we cite the following statement, which we will use immediately afterwards:

**Proposition 2.5.** ([3], 1.57) *Let  $X, Y$  be two diffeological spaces, and let  $U$  be a domain of some  $\mathbb{R}^n$ . A map  $p : U \rightarrow C^\infty(X, Y)$  is a plot for the functional diffeology of  $C^\infty(X, Y)$  if and only if the induced map  $U \times X \rightarrow Y$  acting by  $(u, x) \mapsto p(u)(x)$  is smooth.*

We are now ready to prove the following lemma:

**Lemma 2.6.** *The following statements hold:*

1. *The map  $\tilde{F}$  takes values in  $L^\infty(V, L^\infty(V, W))$ ; furthermore, it is smooth with respect to the functional diffeologies of  $B^\infty(V, W)$  and  $L^\infty(V, L^\infty(V, W))$ .*
2. *The map  $\tilde{G}$  is smooth with respect to the functional diffeologies of  $L^\infty(V, L^\infty(V, W))$  and  $B^\infty(V, W)$ .*

3. The maps  $\tilde{F}$  and  $\tilde{G}$  are inverses of each other.

*Proof.* Let us prove 1. We first prove that  $F : V \rightarrow L^\infty(V, W)$  is smooth. Let  $p : U \rightarrow V$  be an arbitrary plot of  $V$ ; by Proposition 2.5, in order to show that  $F \circ p$  is a plot for the functional diffeology on  $L^\infty(V, W)$ , we need to consider the induced map  $U \times V \rightarrow W$  that acts by the assignment  $(u, v') \mapsto (F \circ p)(u)(v') = F(p(u))(v') = f(p(u), v') = f \circ (p \times \text{Id}_V)(u, v')$  and show that it is smooth. Since  $p \times \text{Id}_V$  is obviously a plot for the product diffeology on  $V \times V$  and  $f$  is smooth,  $f \circ (p \times \text{Id}_V)$  is a plot of  $W$ , so it is naturally smooth.

Let us now show that  $\tilde{F} : B^\infty(V, W) \rightarrow L^\infty(V, L^\infty(V, W))$  is smooth; taking  $p : U \rightarrow B^\infty(V, W)$  a plot of  $B^\infty(V, W)$ , we need to show that  $\tilde{F} \circ p$  is a plot of  $L^\infty(V, L^\infty(V, W))$ . To do this, we apply again Proposition 2.5: it suffices to consider the map  $U \times V \rightarrow L^\infty(V, W)$  acting by  $(u, v) \mapsto (\tilde{F} \circ p)(u)(v) = \tilde{F}(p(u))(v) = \text{EV} \circ ((F \circ p) \times \text{Id}_V)(u, v)$ . Having already established that  $F$  is smooth, we can now conclude that  $\tilde{F}$  is smooth as well.

Let us now prove the second point, *i.e.*, that  $\tilde{G} : L^\infty(V, L^\infty(V, W)) \rightarrow B^\infty(V, W)$  is smooth, *i.e.*, taking an arbitrary plot  $p : U \rightarrow L^\infty(V, L^\infty(V, W))$ , we need to show that  $\tilde{G} \circ p$  is a plot of  $B^\infty(V, W)$ . Applying again Proposition 2.5, we consider the map  $U \times (V \times V) \rightarrow W$  defined by  $(u, (v, v')) \mapsto (\tilde{G} \circ p)(u)(v, v') = (\text{EV} \circ (p(u) \times \text{Id}_V))(v, v') = (\text{EV} \circ (p \times \text{Id}_{V \times V}))(u, (v, v'))$ , which allows us to conclude that the map is smooth, and therefore  $\tilde{G} \circ p$  is a plot of  $B^\infty(V, W)$ ; whence the conclusion.

To conclude, we observe that the third point follows immediately from the definitions of the two maps.  $\square$

We now get the desired conclusion, which does mimick what happens in the usual linear algebra case:

**Theorem 2.7.** *Let  $V$  and  $W$  be two diffeological vector spaces, let  $B^\infty(V, W)$  be the space of all smooth bilinear maps  $V \times V \rightarrow W$  considered with the functional diffeology, and let  $L^\infty(V, L^\infty(V, W))$  be the space of all smooth linear maps  $V \rightarrow L^\infty(V, W)$  endowed, it as well, with the functional diffeology. Then the spaces  $B^\infty(V, W)$  and  $L^\infty(V, L^\infty(V, W))$  are diffeomorphic as diffeological vector spaces.*

*Proof.* The desired diffeomorphism as diffeological spaces is given by the maps  $\tilde{F}$  and  $\tilde{G}$  of Lemma 2.6. It remains to note that these two maps are also linear (actually, as vector spaces maps they coincide with the usual constructions), and that all the functional diffeologies involved are vector space diffeologies.  $\square$

### 3 The diffeological dual

In this section we consider the diffeological dual (as a matter of curiosity, we also comment on an alternative notion, only to show why the existing one is better); this discussion stems from the previous section, which illustrates how the function spaces of linear maps change upon introducing the requirement of diffeological smoothness.

#### 3.1 The dual as the set of smooth linear maps

We now comment on the already-standard notion of the diffeological dual  $V^* = L^\infty(V, \mathbb{R})$  ([6], [7]). Note that, as has already been observed in [7], even in the finite-dimensional case the diffeological dual might be much smaller than the usual one, which corresponds precisely to the case  $L^\infty(V, \mathbb{R}) < L(V, \mathbb{R})$  (also illustrated by our Example 2.1). A simple but natural question to ask at this point is, suppose that  $V$  is a finite-dimensional diffeological vector space such that  $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$  as vector spaces; does this imply that  $V$  is also diffeomorphic to  $V^* = L^\infty(V, \mathbb{R})$ ? The following proposition provides a positive answer to this question.<sup>4</sup>

**Proposition 3.1.** *Let  $V$  be a finite-dimensional diffeological vector space such that  $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$ , *i.e.*, such that every real-valued linear map from  $V$  is smooth. Then  $V$  is diffeomorphic to  $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$ , *i.e.*, to its diffeological dual.*

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<sup>4</sup>I would like to thank the anonymous referee for pointing out a much simpler proof of this statement.

*Proof.* Let  $\{v_1, \dots, v_n\}$  be a basis of  $V$ , and let  $v_i^*$  be the corresponding dual basis in the usual sense (*i.e.*, if  $v = \alpha_1 v_1 + \dots + \alpha_n v_n$  then  $v_i^*(v) = \alpha_i$ ). Observe that each  $v_i^*$  is smooth by assumption.

Consider the linear map  $v \mapsto \sum_{i=1}^n v_i^*(v) v_i^*$ ; this is a map from  $V$  that takes values in  $V^*$  by the already-made observation. Since  $V$  is finite-dimensional, this map is obviously bijective, with the inverse given by  $v^* \mapsto v^*(v_i) v_i$ . It remains to observe that this inverse is also smooth, and so the map indicated is indeed a diffeomorphism of diffeological vector spaces  $V$  and  $V^*$ .  $\square$

In the following section we briefly consider another possibility for defining the diffeological dual.

### 3.2 The dual as the set of linear maps with pushforward diffeology

In this subsection, we comment on what happens if (in the finite-dimensional case only) we consider a possible alternative to the notion of the diffeological dual. This alternative is to consider the usual space of all linear maps  $V \rightarrow \mathbb{R}$  endowed with the pushforward diffeology relative to the standard isomorphism of  $V$  with its usual dual:

- let  $V$  be a finite-dimensional diffeological vector space, and let  $\hat{V}^*$  be the usual vector space dual of  $V$  endowed with the following diffeology: choose an isomorphism  $\hat{f} : V \rightarrow \hat{V}^*$  and denote by  $\mathcal{D}_{\hat{f}}$  the pushforward<sup>5</sup> of the diffeology of  $V$  by the map  $\hat{f}$ .

Such definition presents the obvious question of being well-posed, *i.e.*, whether the diffeology obtained depends on the choice of the isomorphism.

**Lemma 3.2.** *Let  $V$  be a finite-dimensional diffeological vector space, let  $\hat{f} : V \rightarrow \hat{V}^*$  and  $\hat{g} : V \rightarrow \hat{V}^*$  be two vector space isomorphisms of  $V$  with its dual, and let  $\mathcal{D}_{\hat{f}}$  and  $\mathcal{D}_{\hat{g}}$  be the corresponding pushforward diffeologies. Then  $\mathcal{D}_{\hat{f}} = \mathcal{D}_{\hat{g}}$ .*

*Proof.* It is sufficient to show that the composition map  $g \circ f^{-1} : (\hat{V}^*, \mathcal{D}_{\hat{f}}) \rightarrow (\hat{V}^*, \mathcal{D}_{\hat{g}})$  is smooth with respect to the pushforward diffeology. Let  $p : U \rightarrow (\hat{V}^*, \mathcal{D}_{\hat{f}})$  be a plot of  $(\hat{V}^*, \mathcal{D}_{\hat{f}})$ ; we need to show that  $(g \circ f^{-1}) \circ p$  is also a plot, of  $(\hat{V}^*, \mathcal{D}_{\hat{g}})$ . By definition of the pushforward diffeology,  $p$  being a plot of  $(\hat{V}^*, \mathcal{D}_{\hat{f}})$  implies that (up to passing to a smaller neighbourhood) there exists a plot  $p' : U \rightarrow V$  of  $V$  such that  $p = p' \circ \hat{f}$ . Since  $\hat{f}$  is invertible, we can write now  $(g \circ f^{-1}) \circ p = g \circ (f^{-1} \circ p) = g \circ p'$ ; and the latter map is by definition a plot of the pushforward diffeology on  $\hat{V}^*$  by  $g$  (that is, it is a plot of  $(\hat{V}^*, \mathcal{D}_{\hat{g}})$ ), whence the conclusion.  $\square$

Since not all vector spaces are isomorphic to their duals, the limitations of this approach are obvious. A still more important reason is that the functional diffeology is far more natural for the dual, as demonstrated by the heavy use of its properties in various proofs (including ones in the present paper). Finally, the example in the next section shows that a very natural property of dual spaces does *not* hold for  $\hat{V}^*$ .

### 3.3 The dual map

In this section we speak of the dual maps of linear maps; recall that, given two vector spaces  $V, W$ , and a linear map  $f : V \rightarrow W$  between them, then the dual map is  $f^* : W^* \rightarrow V^*$  acting by  $f^*(g)(v) = g(f(v))$ . If now  $V$  and  $W$  are diffeological vector spaces, and  $f \in L^\infty(V, W)$ , there is the obvious question whether the corresponding  $f^*$  is a smooth map. We provide an explicit proof that the answer is positive for  $V^*$  (this is implicit in [7], see p.7), but *a priori* it is negative for  $\hat{V}^*$ .

**Proposition 3.3.** *Let  $V, W$  be two diffeological vector spaces, and let  $f : V \rightarrow W$  be a smooth linear map. Let  $f^* : W^* \rightarrow V^*$  be the dual map between the diffeological duals,  $f^*(g)(v) = g(f(v))$ . Then  $f^*$  is smooth.*

<sup>5</sup>Even if in the case of an isomorphism it does not make a difference, we mention that instead of a pushforward of the diffeology of  $V$  we could speak of its pullback by the inverse isomorphism.

*Proof.* Let  $p$  be a plot of  $W^*$ ; we need to show that  $f^* \circ p$  is a plot of  $V^*$ . The diffeology of  $W^*$  being functional, by Proposition 2.5,  $p$  being a plot is equivalent to the smoothness of the map  $\psi : U \times W \rightarrow \mathbb{R}$  acting by  $\psi(u, w) = p(u)(w)$ ; now, for  $\psi$  to be smooth, we must have for any plot  $(p_1, p_W) : U' \rightarrow U \times W$  (where  $p_1$  and  $p_W$  are plots of  $U'$  and  $W$  respectively) that  $\psi \circ (p_1, p_W)$  is a plot of the standard  $\mathbb{R}$ . For future use, let us write explicitly that  $(\psi \circ (p_1, p_W))(u') = p(p_1(u'))(p_W(u'))$ .

Now, to prove that  $f^* \circ p$  is a plot of  $V^*$ , we need to show that the map  $\varphi : U \times V \rightarrow \mathbb{R}$  given by  $\varphi(u, v) = p(u)(f(v))$  is smooth, that is, that for any plot  $(p_1, p_V) : U' \rightarrow U \times V$  (where  $p_1$  and  $p_V$  are plots of  $U'$  and  $V$  respectively) the composition  $\varphi \circ (p_1, p_V)$  is a plot of  $\mathbb{R}$ . Writing explicitly  $(\varphi \circ (p_1, p_V))(u') = p(p_1(u'))(f(p_V(u')))$  and observing that  $f$  being smooth and  $p_V$  being a plot of  $V$ , we get that  $f \circ p_V$  is a plot of  $W$ , so setting  $p_W = f \circ p_V$ , we deduce immediately the desired conclusion from the analogous expression for  $\psi$  (smooth by assumption) and  $p_W$ .  $\square$

We now briefly describe an example that shows that the dual map  $f^* : \hat{W}^* \rightarrow \hat{V}^*$  may not be smooth.

**Example 3.4.** Let  $V$  be  $\mathbb{R}^n$  with the fine diffeology, and let  $W$  be  $\mathbb{R}^n$  with the coarse diffeology. Observe that this implies that  $\hat{V}^*$  and  $\hat{W}^*$ , being pullbacks, also have, respectively, the fine and the coarse diffeology. Let  $f : V \rightarrow W$  be any linear map (it is automatically smooth); then  $f^*$  is a map from  $\mathbb{R}^n$  with the coarse diffeology to the one with the fine diffeology. It suffices to choose, as a plot  $p$  of  $\hat{W}^*$ , any non-smooth map to  $\mathbb{R}^n$ , to get that  $f^* \circ p$  is not a plot of  $\hat{V}^*$ , thus disproving the smoothness of  $f^*$ .

## 4 The tensor product

In this section we discuss the definition (as given in [7]), and the relative properties, of the tensor product; we speak mostly of the case of two factors, given that the extension to the case of more than two spaces is *verbatim*.

**The tensor product of maps** Let us consider two (smooth) linear maps between diffeological vector spaces,  $f : V \rightarrow V'$  and  $g : W \rightarrow W'$ . As usual, we have the tensor product map  $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ , defined by  $(f \otimes g)(\sum v_i \otimes w_i) = \sum f(v_i) \otimes g(w_i)$ . We observe that  $f \otimes g$  is a smooth map (with respect to the tensor product diffeologies on  $V \otimes W$  and  $V' \otimes W'$ ) due to the properties of the product and the quotient diffeologies.

**The tensor product and the direct sum** Let  $V_1, V_2, V_3$  be vector spaces; recall that in the usual linear algebra the tensor product is distributive with respect to the direct sum, *i.e.*:

$$V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3),$$

via a canonical isomorphism, which we denote by  $T_{\otimes, \oplus}$ . Now, if  $V_1, V_2, V_3$  are diffeological vector spaces, then so are  $V_1 \otimes (V_2 \oplus V_3)$  and  $(V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$ . It turns out, as already mentioned in [7], Remark 3.9 (2), that the standard isomorphism between these spaces is also a diffeomorphism. Below we provide an explicit proof of that statement.

**Lemma 4.1.** ([7]) *Let  $V_1, V_2, V_3$  be diffeological vector spaces, and let  $T_{\otimes, \oplus} : V_1 \otimes (V_2 \oplus V_3) \rightarrow (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$  be the standard isomorphism. Then  $T_{\otimes, \oplus}$  is smooth.*

*Proof.* By the properties of the quotient diffeology, it is sufficient to show that the covering map  $\tilde{T}_{\otimes, \oplus} : V_1 \times (V_2 \oplus V_3) \rightarrow (V_1 \times V_2) \oplus (V_1 \times V_3)$  is smooth. Let  $p : U \rightarrow V_1 \times (V_2 \oplus V_3)$  be a plot; we must show that  $\tilde{T}_{\otimes, \oplus} \circ p$  is a plot for  $(V_1 \times V_2) \oplus (V_1 \times V_3)$ . Let  $\pi_1 : V_1 \times (V_2 \oplus V_3) \rightarrow V_1$  and  $\pi_{2,3} : V_1 \times (V_2 \oplus V_3) \rightarrow (V_2 \oplus V_3)$  be the natural projections; observe that by definition of the product diffeology,  $\pi_{2,3}$  writes (at least locally) as  $\pi_{2,3} = (p_2, p_3)$ , where  $p_2$  is a plot of  $V_2$  and  $p_3$  is a plot of  $V_3$ .

Write now  $\tilde{T}_{\otimes, \oplus} \circ p$  as  $\tilde{T}_{\otimes, \oplus} \circ p = (p', p'')$ ; observe that  $p' = (\pi_1 \circ p, p_2)$ , while  $p'' = (\pi_1 \circ p, p_3)$ . These are plots for the sum diffeology on  $(V_1 \times V_2) \oplus (V_1 \times V_3)$ , hence the conclusion.  $\square$



**The tensor product  $V \otimes W$  as a function space** Recall that in the usual linear algebra context the tensor product of two finite-dimensional vector spaces  $V \otimes W$  is isomorphic to the spaces  $L(V^*, W)$ , the space of linear maps  $V^* \rightarrow W$ , and  $L(W^*, V)$ , the space of linear maps  $W^* \rightarrow V$ , via isomorphisms given by:

- for  $f \in V^*$ ,  $v \in V$ , and  $w \in W$  we set  $(v \otimes w)(f) = f(v)w$ , extending by linearity;
- for  $g \in W^*$ ,  $v \in V$ , and  $w \in W$  we set  $(v \otimes w)(g) = g(w)v$ , extending by linearity.

The question that we consider now is whether these isomorphisms continue to exist if all spaces we consider are finite-dimensional diffeological vector spaces, all linear maps are smooth, and all function spaces are endowed with their functional diffeologies. The observations made regarding the frequently substantial difference between a diffeological vector space  $V$  and its diffeological dual  $V^*$  suggest that we consider again one of our examples.

**Example 4.2.** Let  $V = \mathbb{R}^n$  for  $n \geq 2$  with the coarse diffeology, and let  $W = \mathbb{R}$  with the standard diffeology. Then, as shown in Example 2.1, the diffeological dual of  $V$  is trivial:  $V^* = \{0\}$ ; this obviously implies that  $L^\infty(V^*, W) = \{0\}$ . Recall also that, the diffeology of  $W$  being fine, its dual is isomorphic to  $W$ , so we have  $W \cong W^* \cong \mathbb{R}$ ; furthermore, as it occurs for all fine diffeological vector spaces (see [3]), we have  $L^\infty(W^*, V) = L(W^*, V) \cong V$ .

Since the total space of the diffeological tensor product  $V \otimes W$  is the same as that of the usual tensor product, it is isomorphic to  $V$ . Therefore there is **not** an isomorphism between  $V \otimes W$  and  $L^\infty(V^*, W)$ , the two spaces being different as sets. On the other hand,  $L^\infty(W^*, V)$  and  $V \otimes W$  are isomorphic as usual vector spaces; it is easy to see that they are also diffeomorphic (this follows from the fact that  $V$  has the coarse diffeology<sup>6</sup>).

The example just made shows that in general, at least one of these classical isomorphisms might fail to exist (and at a very basic level). We may wish however to see what could be kept of the standard isomorphisms, in the sense that the two maps  $V \otimes W \rightarrow L(V^*, W)$  and  $V \otimes W \rightarrow L(W^*, V)$  are still defined; we wonder if their ranges consist of smooth maps and, if so, whether they are smooth.

**Proposition 4.3.** Let  $V, W$  be two finite-dimensional diffeological vector spaces. Then:

1. If  $\hat{F} : V \otimes W \rightarrow L(V^*, W)$  is the map defined, via linearity, by  $v \otimes w \mapsto [\hat{F}(v \otimes w)(f) = f(v)w]$  then  $\hat{F}$  takes values in  $L^\infty(V^*, W)$ . Furthermore, as a map  $V \otimes W \rightarrow L^\infty(V^*, W)$  between diffeological spaces, it is smooth;
2. If  $\hat{G} : V \otimes W \rightarrow L(W^*, V)$  is the map defined, via linearity, by  $v \otimes w \mapsto [\hat{G}(v \otimes w)(g) = g(w)v]$  then  $\hat{G}$  takes values in  $L^\infty(W^*, V)$ . Furthermore, as a map  $V \otimes W \rightarrow L^\infty(W^*, V)$  between diffeological spaces, it is smooth.

*Proof.* Let us prove 1. We need to show that  $\hat{F}$  is a smooth map that takes values in  $L^\infty(V^*, W)$ . To prove the latter, it is enough to show that  $\hat{F}(v \otimes w)$  is smooth, for any  $v \in V$  and  $w \in W$ . Let us fix  $v \in V$  and  $w \in W$ ; we need to show that for any plot  $p : U \rightarrow V^*$  the composition  $\hat{F}(v \otimes w) \circ p$  is a plot of  $W$ . Writing explicitly  $(\hat{F}(v \otimes w) \circ p)(u) = \hat{F}(v \otimes w)(p(u)) = p(u)(v)w$ , we recall that any constant map on a domain is a plot for any diffeology, so the map  $c_w : U \rightarrow W$  that sends everything in  $w$  is a plot of  $W$ . Finally, the map  $(u, v) \mapsto p(u)(v)$  is a smooth map to  $\mathbb{R}$ , by Proposition 2.5 and because  $p$  is a plot of  $V^* = L^\infty(V, \mathbb{R})$  whose diffeology is functional; recalling that multiplication by scalar is smooth for any diffeological vector space, we get the conclusion.

Let us now prove that  $\hat{F}$  is a smooth map  $V \otimes W \rightarrow L^\infty(V^*, W)$ ; by Proposition 2.5 we need to prove that the induced map  $V^* \times U \rightarrow W$  is smooth.<sup>7</sup> This map acts by sending each  $(f, u)$  (where  $f \in V^*$ ) to  $(\hat{F} \circ p)(u)(f)$  and so it writes as  $(f, u) \mapsto (\text{EV}_{V^*} \otimes \text{Id}_W)(\text{Id}_{V^*} \times p)(f, u)$ ; the diffeology of  $V^*$  being functional, the evaluation map is smooth, therefore so is  $\hat{F} \circ p$ , whence the conclusion.

The proof of 2 is completely analogous, so we omit it.  $\square$

<sup>6</sup>Consider the obvious map  $F : V \rightarrow L(\mathbb{R}, V) = L^\infty(\mathbb{R}, V)$  given by  $F(v)(x) = xv$ ; it is obviously bijective, and it is smooth by Proposition 2.5. Indeed, for any plot  $p : U \rightarrow V$  we need that  $F \circ p$  be a plot, which is equivalent to the map  $U \times \mathbb{R} \rightarrow V$  given by  $(u, x) \mapsto (F \circ p)(u)(x) = xp(u)$  being smooth. But simply due to the fact that it is a map in  $V$ , that has the coarse diffeology, it is a plot of it, so the conclusion.

<sup>7</sup>Note the change in the order of factors, for formal purposes.

**Observation 4.4.** *Example 4.2 also illustrates that, in general, there is **not** an analogue of the classical isomorphism  $V^* \otimes V \cong L^\infty(V, V)$ : it suffices to consider the same  $V$ , that is,  $\mathbb{R}^n$  with the coarse diffeology. Then the product on the left is the trivial space,  $V^*$  being the trivial space, whereas the space on the right consists of all linear maps  $V \rightarrow V$  (since the coarse diffeology includes any map into  $V$ , all of these maps are automatically smooth).*

**Tensor product of duals and the dual of a tensor product** Recall, once again, that for usual vector spaces there is a standard isomorphism  $V^* \otimes W^* \cong (V \otimes W)^*$ ; we are now interested in the question whether the existence of this isomorphism extends to the diffeological context, *i.e.*, whether the corresponding map is smooth. This standard isomorphism  $V^* \otimes W^* \rightarrow (V \otimes W)^*$ , which in this paragraph we denote by  $F$ , is defined by:

$$F\left(\sum_i f_i \otimes g_i\right)\left(\sum_j v_j \otimes w_j\right) = \sum_{i,j} f_i(v_j)g_i(w_j).$$

The first thing that we need to check is whether it does take values in  $(V \otimes W)^*$ , that is, if, fixed some  $f \otimes g \in V^* \otimes W^*$ ,<sup>8</sup> it actually defines a smooth (and not just linear) map  $V \otimes W \rightarrow \mathbb{R}$ .

**Lemma 4.5.** *Let  $V, W$  be diffeological vector spaces, and let  $f \in V^*, g \in W^*$ . Then the map  $F(f \otimes g) : V \otimes W \rightarrow \mathbb{R}$  is smooth.*

*Proof.* By Proposition 2.5 we need to check that for any plot  $p : U \rightarrow V \otimes W$  the composition  $F(f \otimes g) \circ p$  is a smooth map  $U \rightarrow \mathbb{R}$ . Recall that locally (so we assume that  $U$  is small enough)  $p$  writes as a composition  $p = \pi_\otimes \circ \tilde{p}$ , where  $\pi_\otimes$  is the natural projection  $V \times W \rightarrow V \otimes W$  and  $\tilde{p} : U \rightarrow V \times W$  is a plot for the product diffeology; furthermore,  $\tilde{p}$  writes as  $\tilde{p} = (p_V, p_W)$ , where  $p_V$  is a plot of  $V$  and  $p_W$  is a plot of  $W$ . Therefore we have  $(F(f \otimes g) \circ p)(u) = f(p_V(u))g(p_W(u))$ , that is,  $F(f \otimes g) \circ p$  is the usual product in  $\mathbb{R}$  of two maps,  $f \circ p_V$  and  $g \circ p_W$ . Now,  $f$  being smooth by its choice and  $p_V$  being a plot of  $V$ , their composition  $f \circ p_V$  is a smooth map in  $\mathbb{R}$ . The same holds also for  $g \circ p_W$ ; the product of two smooth maps being smooth, we get the desired conclusion.  $\square$

By the lemma just proven,  $F$  is an injective linear map from the tensor product of the diffeological duals  $V^*, W^*$  into the diffeological dual of the tensor product  $V \otimes W$ . We should check next whether it is smooth.

**Proposition 4.6.** *Let  $V, W$  be diffeological vector spaces, and let  $F : V^* \otimes W^* \rightarrow (V \otimes W)^*$  be the already defined map between the diffeological duals. Then  $F$  is smooth.*

*Proof.* For the map  $F$  to be smooth, it is required that, for any plot  $p : U \rightarrow V^* \otimes W^*$  the composition  $F \circ p$  be a plot of  $(V \otimes W)^*$ , which is equivalent to the smoothness of the map  $\Phi : U \times (V \otimes W) \rightarrow \mathbb{R}$  such that  $\Phi(u, \sum v_j \otimes w_j) = F(p(u))(\sum v_j \otimes w_j)$ . The map  $\Phi$  being smooth is in turn equivalent to the following: for any map  $(p_U, p_{V \otimes W}) : U' \rightarrow U \times (V \otimes W)$  such that  $p_U : U' \rightarrow U$  is smooth and  $p_{V \otimes W} : U' \rightarrow V \otimes W$  is a plot of  $V \otimes W$  the composition  $\Phi \circ (p_U, p_{V \otimes W})$  is a smooth map  $U' \rightarrow \mathbb{R}$ . We write explicitly:

$$(\Phi \circ (p_U, p_{V \otimes W}))(u') = F((p \circ p_u)(u'))(p_{V \otimes W}(u')).$$

By definition of the quotient diffeology, for  $p : U \rightarrow V^* \otimes W^*$  is a plot of its target space, we must have that for  $U$  small enough  $p$  lifts to a smooth map  $\tilde{p} : U \rightarrow V^* \times W^*$ , that is,  $p = \pi_{V^* \otimes W^*} \circ \tilde{p}$ , where  $\pi_{V^* \otimes W^*}$  is the natural projection; moreover,  $\tilde{p}$  writes as  $\tilde{p} = (p_{V^*}, p_{W^*})$ , where  $p_{V^*}$  is a plot of  $V^*$  and  $p_{W^*}$  is a plot of  $W^*$ . Recall that  $p_{V^*}$  being a plot of  $V^*$  means that the map  $\varphi_V : U \times V \rightarrow \mathbb{R}$  given by  $\varphi_V(u, v) = p_{V^*}(u)(v)$  is smooth; accordingly,  $p_{W^*}$  being a plot of  $W^*$  means that the map  $\varphi_W : U \times W \rightarrow \mathbb{R}$  given by  $\varphi_W(u, w) = p_{W^*}(u)(w)$  is smooth. Furthermore, for  $p_{V \otimes W}$  be a plot, we should have for  $U$  small enough,  $p_{V \otimes W}$  lifts to  $\tilde{p}_{V \times W}$ , a plot of  $V \times W$ , that is,  $p_{V \otimes W}$  writes as  $p_{V \otimes W} = \pi_\otimes \circ \tilde{p}_{V \times W}$  for the appropriate natural projection  $\pi_\otimes$ ; furthermore,  $\tilde{p}_{V \times W}$  writes as  $\tilde{p}_{V \times W} = (p_V, p_W)$ , where  $p_V$  is a plot of  $V$  and  $p_W$  is a plot of  $W$ .

<sup>8</sup>Extending by linearity is smooth by definition of a diffeological vector space.

Assume now that the domain  $U$  is small enough so that all of the above be valid; then we can write, by definition of  $F$ , that

$$(\Phi \circ (p_U, p_{V \otimes W}))(u') = p_{V^*}(u')(p_V(u')) \cdot p_{W^*}(u')(p_W(u')) = \text{EV}(p_{V^*}, p_V)(u') \cdot \text{EV}(p_{W^*}, p_W)(u'),$$

where  $(p_{V^*}, p_V) : U' \rightarrow V^* \times V$  and  $(p_{W^*}, p_W) : U' \rightarrow W^* \times W$  are the obvious maps. By definition of the product diffeology they are plots for, respectively,  $V^* \times V$  and  $W^* \times W$ ; furthermore, by definition of the functional diffeology  $\text{EV}$  is smooth. It follows that  $\Phi \circ (p_U, p_{V \otimes W}) : U' \rightarrow \mathbb{R}$  writes as the product of two smooth maps  $U' \rightarrow \mathbb{R}$ ; the diffeology of  $\mathbb{R}$  being the standard one, these maps are smooth in the usual sense, hence so is their product. This implies that  $\Phi$  is a smooth map, therefore  $F$  is smooth, and the Proposition is proven.  $\square$

We are now ready to prove the following statement:

**Theorem 4.7.** *Let  $V, W$  be two finite-dimensional diffeological vector spaces. Then  $F : V^* \otimes W^* \rightarrow (V \otimes W)^*$  is a diffeomorphism.*

*Proof.* It remains to check that  $F$  is surjective with smooth inverse, *i.e.*, that for any smooth linear map  $f : V \otimes W \rightarrow \mathbb{R}$  its pre-image  $F^{-1}(f)$  actually belongs to the tensor product of the diffeological duals. By definition of  $F$ , it is sufficient to observe that  $f$  being smooth means that for any plot  $p : U \rightarrow V \otimes W$  the composition  $f \circ p : U \rightarrow \mathbb{R}$  is a usual smooth map; furthermore, for  $U$  small enough  $p$  writes as  $p = \pi \circ (p_V, p_W)$ , where  $\pi : V \times W \rightarrow V \otimes W$  is the natural projection,  $p_V : U \rightarrow V$  is a plot of  $V$ , and  $p_W : U \rightarrow W$  is a plot of  $W$ , hence  $f \circ p$  actually writes as  $(f \circ p)(u) = f(p_V(u) \otimes p_W(u))$ . Note that  $F^{-1}(f)$  writes as  $F^{-1}(f) = \sum f_i \otimes g_i$  with  $f_i$  belonging to the usual dual of  $V$ , and  $g_i$  belonging to the usual dual of  $W$ ; we obtain that  $(F^{-1}(f) \circ (p_V, p_W))(u) = \sum (f_i \circ p_V)(u)(g_i \circ p_W)(u)$ , and we can draw the desired conclusion by choosing the appropriate plots  $p_V$  and  $p_W$ .  $\square$

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